

# Finite-Difference Methods for Nonlinear Hyperbolic Systems. II

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**Introduction.** In [3] the authors introduced several schemes for the numerical integration of nonlinear hyperbolic systems. In this paper these schemes are extended to solve the nonlinear systems

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}(u, x, t) = z(u, x, t),$$
$$\frac{\partial u}{\partial t} + A(u, x, t) \frac{\partial u}{\partial x} = z(u, x, t)$$

and the corresponding problems in two-space dimensions. Also a procedure is developed whereby the extra boundary data required by these schemes can be introduced in a smooth manner.

**1. Explicit One-Space Dimensional Scheme.** Consider the first-order hyperbolic system

$$(1.1) \quad \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}(u, x, t) = z(u, x, t)$$

where  $u$  is an unknown vector function of  $x$  and  $t$  and  $f$  and  $z$  are vector functions of the components of  $u$ , and of  $x$  and  $t$ . We shall be concerned with the solution of this problem in the region

$$0 \leq x \leq \alpha, \quad t \geq 0,$$

and will assume initial data  $u(x, 0) = u_0(x)$  and boundary data  $u(0, t) = u_1(t)$ ,  $t > 0$ . This problem is only properly posed if the Jacobian matrix of the components of  $f$  with respect to the components of  $u$  is positive definite. We have made this assumption to simplify the analysis.

If differentiation is carried out in (1.1) the equation

$$(1.2) \quad \frac{\partial u}{\partial t} + A(u, x, t) \frac{\partial u}{\partial x} = z(u, x, t)$$

is obtained, where  $z$  in (1.2) is not necessarily the same as in (1.1), and where  $A(u, x, t)$ , the Jacobian matrix of  $f$  with respect to  $u$ , is positive definite. Whilst we can always derive an equation of the form (1.2) from (1.1), the process is not always reversible. We therefore will consider (1.1) and (1.2) separately. The definitions and notation will be as in [3]. We therefore have

$$u(x_i, t_m) = u(ih, mk) = u_i^m = u_m, \quad i = 0, \dots, N, m = 0, 1, \dots,$$

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$$H_x u_m = u_{i+1}^m - u_{i-1}^m, \\ p = k/h,$$

where  $k$  and  $h$  are the mesh spacings in the time and space directions respectively.

The proposed scheme to solve (1.1) then takes the form

$$(1.3) \quad u_{m+1}^* = \hat{u}_m - ap(H_x f_m - 2hz_m),$$

$$(1.4) \quad u_{m+1} = u_m - p[bH_x f_m + cH_x f_{m+1}^* + eH_x \bar{f}_{m+1} + 2h(dz_{m+1}^* + s\bar{z}_{m+1} + qz_m)],$$

where  $a, b, c, e, d, s, q$  are constants to be determined and

$$\hat{u}_m = \frac{1}{2}(u_{i+1}^m + u_{i-1}^m), \\ f_{m+1}^* = f(u_{m+1}^*, x_i, t_m), \quad z_{m+1}^* = z(u_{m+1}^*, x_i, t_m), \\ \bar{f}_{m+1} = f(u_m, x_i, t_{m+1}), \quad \bar{z}_{m+1} = z(u_m, x_i, t_{m+1}).$$

The scheme (1.3) is a first-order approximation to (1.1) evaluated at time level  $(m + 2a)k$ . We require that the overall scheme (1.3) and (1.4) (on elimination of  $u_{m+1}^*$ ) be a second-order correct approximation to (1.1).

By substituting for  $u_{m+1}^*$  from (1.3) into (1.4) and expanding by Taylor's theorem we obtain

$$(1.5) \quad u_{m+1} = u - 2ph \left[ (b + c + e) \frac{\partial f}{\partial x} + (d + s + q)z \right] \\ + p^2 h^2 \left[ 4ac \frac{\partial}{\partial x} \left( A \left( \frac{\partial f}{\partial x} - z \right) \right) + 4ad \left( \frac{\partial f}{\partial x} - z \right) \frac{\partial z}{\partial u} \right. \\ \left. - 2e \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial t} \right) - 2s \frac{\partial z}{\partial t} \right] + O(h^3),$$

where  $u = u_m$ , and where we have used (1.1). An expansion of  $u_{m+1}$  in terms of  $u$  ( $\equiv u_m$ ) and its derivatives yield

$$(1.6) \quad u_{m+1} = u + ph \left( z - \frac{\partial f}{\partial x} \right) + \frac{p^2 h^2}{2} \\ \left[ \frac{\partial z}{\partial t} + \frac{\partial z}{\partial u} \left( z - \frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial t} + A \left( z - \frac{\partial f}{\partial x} \right) \right) \right] + O(h^3),$$

where use has again been made of (1.1), and where

$$\frac{\partial \theta}{\partial t} = \lim_{\Delta t \rightarrow 0} \left\{ \frac{\theta(u(x, t), x, t + \Delta t) - \theta(u(x, t), x, t)}{\Delta t} \right\}.$$

Comparing coefficients in (1.5) and (1.6), we see that for a second-order correct scheme we require that the equations

$$2(b + c + e) = 1, \quad 2(d + s + q) = -1, \\ 4ac = \frac{1}{2}, \quad 4ad = -\frac{1}{2}, \quad 2e = \frac{1}{2}, \quad 2s = -\frac{1}{2}$$

must be satisfied. These equations have the solution

$$c = -d = 1/8a, \quad b = -q = \frac{1}{4}(1 - 1/2a), \quad e = -s = \frac{1}{4}$$

in terms of the parameter  $a$ .

The scheme (1.3), (1.4) can then be written as

$$\begin{aligned}
 u_{m+1}^* &= \hat{u}_m - ap(H_x f_m - 2hz_m), \\
 (1.7) \quad u_{m+1} &= u_m - p/4[(1 - 1/2a)(H_x f_m - 2hz_m) \\
 &\quad + (1/2a)(H_x f_{m+1}^* - 2hz_{m+1}^*) + (H_x \bar{f}_{m+1} - 2h\bar{z}_{m+1})].
 \end{aligned}$$

A similar analysis for Eq. (1.2) yields the scheme

$$\begin{aligned}
 u_{m+1}^* &= \hat{u}_m - ap(A_m H_x u_m - 2hz_m), \\
 (1.8) \quad u_{m+1} &= u_m - p/4[(1 - 1/2a)(A_m H_x u_m - 2hz_m) + (1/2a) \\
 &\quad \times (A_{m+1}^* H_x u_{m+1}^* - 2hz_{m+1}^*) + (\bar{A}_{m+1} H_x u_m - 2h\bar{z}_{m+1})],
 \end{aligned}$$

where  $A_{m+1}^* = A(u_{m+1}^*, x_i, t_m)$  and  $\bar{A}_{m+1} = A(u_m, x_i, t_{m+1})$ .

In [3], the authors derived for the simpler equations

$$\begin{aligned}
 \partial u / \partial t + \partial f(u) / \partial x &= 0, \\
 \partial u / \partial t + A(u) \partial u / \partial x &= 0
 \end{aligned}$$

an iterative scheme based on the analogues of (1.7) and (1.8). This process can be derived in a similar manner for Eqs. (1.1) and (1.2). We briefly state the scheme for Eq. (1.1), having put  $a = \frac{1}{2}$ .

$$\begin{aligned}
 u_{m+1}^* &= \hat{u}_m - p/2[H_x f_m - 2hz_m], \\
 u_{m+1}^{(j+1)} &= u_m - p/4[(H_x f_{m+1}^{(j)} - 2hz_{m+1}^{(j)}) + (H_x \bar{f}_{m+1} - 2h\bar{z}_{m+1})], \quad j = 0, 1, \dots
 \end{aligned}$$

where

$$f_{m+1}^{(0)} = f_{m+1}^* \quad \text{and} \quad z_{m+1}^{(0)} = z_{m+1}^*.$$

Although this scheme gave reasonable results in numerical examples we shortly propose a modification of (1.7) which appears to give comparable accuracy with much less computation.

Whereas the differential problem (1.1) is well posed with boundary data only given on  $x = 0$ , the finite-difference scheme (1.7) requires in addition data on the line  $x = \alpha$ .

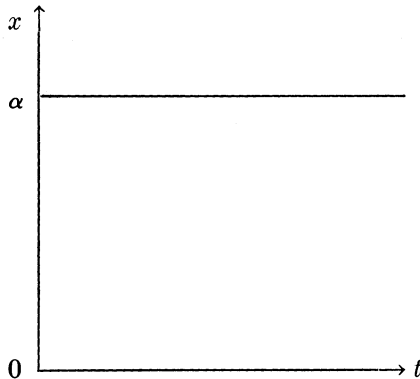


FIGURE 1

As the experiments of Richtmyer and Morton [5], Gary [2] and Parter [4] have shown, when a numerical method is implemented, considerable care has to be taken when inserting such extra data.

In [3] we used the above iteration technique together with extra data derived from the theoretical solution to the problem. An alternative technique is now proposed which is relatively easy to implement in all situations. [We have restricted ourselves to the case  $A(u, x, t) > 0$  to simplify the following analysis. It may be carried through in general.]

There are two boundaries on which data requires to be introduced in order to use (1.7), namely the  $x = 0$  and  $x = \alpha$  boundaries, where  $\alpha = Nh$ . Since we have assumed the matrix  $A$  to be positive definite, the differential problem is well posed with boundary data given on  $x = 0$ . If theoretical data is imposed on  $x = \alpha$  then the problem is overdetermined. We shall avoid this difficulty by replacing our difference scheme (1.7) for  $x = Nh$  by another scheme which has the *same principal part of truncation error*, but which only involves values of  $u_i^m$  for  $i \leq N$ . We use this scheme to calculate  $u_N^{*m+1}$  and  $u_N^{m+1}$ . Equation (1.7) with operator  $H_x$  written as  $(\Delta_x + \nabla_x)$  and  $\hat{u}_m = \frac{1}{2}(\Delta_x - \nabla_x + 2)u_m$  is of the form

$$\begin{aligned}
 u_{m+1}^* &= \frac{1}{2}(\Delta_x - \nabla_x + 2)u_m - ap[(\Delta_x + \nabla_x)f_m - 2hz_m], \\
 (1.9) \quad u_{m+1} &= u_m - p/4[(1 - 1/2a)\{(\Delta_x + \nabla_x)f_m - 2hz_m\} \\
 &\quad + (1/2a)\{(\Delta_x + \nabla_x)f_{m+1}^* - 2hz_{m+1}^*\} \\
 &\quad + \{(\Delta_x + \nabla_x)\bar{f}_{m+1} - 2h\bar{z}_{m+1}\}].
 \end{aligned}$$

From the definition of difference operators we have

$$(1.10) \quad \Delta_x = \nabla_x + \nabla_x^2 + \nabla_x^3 + O(h^4).$$

We derive our alternative scheme by substituting (1.10) in (1.9) to obtain

$$\begin{aligned}
 (1.11) \quad u_{m+1}^* &= \frac{1}{2}(\nabla_x^2 + 2)u_m - ap[(2\nabla_x + \nabla_x^2)f_m - 2hz_m], \\
 (1.12) \quad u_{m+1} &= u_m - p/4[(1 - 1/2a)\{(2\nabla_x + \nabla_x^2 + \nabla_x^3)f_m - 2hz_m\} \\
 &\quad + (1/2a)\{(2\nabla_x + \nabla_x^2 + \nabla_x^3)f_{m+1}^* - 2hz_{m+1}^*\} \\
 &\quad + \{(2\nabla_x + \nabla_x^2 + \nabla_x^3)\bar{f}_{m+1} - 2h\bar{z}_{m+1}\}].
 \end{aligned}$$

This formula is used only when  $x = \alpha = Nh$ . This scheme has, by virtue of its derivation, the same principal part of truncation error as (1.7). A similar scheme may be derived for (1.8). The actual order of implementation in a problem is as follows:

- (1) the predictor of (1.7) is used for  $i = 1, \dots, N - 1$ ,
- (2) " " " (1.11) " " "  $i = N$ ,
- (3) the corrector of (1.7) " " "  $i = 1, \dots, N - 1$ ,
- (4) " " " (1.12) " " "  $i = N$ .

It is easily seen how this scheme may be extended to problems in which  $A(u, x, t)$  is not positive definite. If we generate the extra data by this technique, then the distribution of errors at any time level is smooth, and no error growth occurs at the  $x = \alpha$  boundary.

**2. Implicit One-Dimensional Scheme.** As an approximation to (1.1) we consider now the implicit scheme

$$(2.1) \quad \begin{aligned} u_{m+1}^* &= \hat{u}_m - ap(H_x f_m - 2hz_m), \\ u_{m+1} &= u_m - p[bH_x f_m + cH_x \bar{A}_{m+1}^* u_{m+1} + dH_x \bar{f}_{m+1} \\ &\quad + 2h(\alpha z_m + \beta z_{m+1}^* + \gamma \bar{z}_{m+1})] \end{aligned}$$

where  $\bar{A}$  is defined by the relation,  $f(u, x, t) = \bar{A}(u, x, t) \cdot u$ .

(Notice that the matrix  $\bar{A}(u, x, t)$  is not defined uniquely by this relation.)

A similar analysis to the explicit case determines the constants  $a, b, c, d, \alpha, \beta, \gamma$  and the resulting scheme, correct to second order, is

$$(2.2) \quad \begin{aligned} u_{m+1}^* &= \hat{u}_m - \frac{1}{2}p(H_x f_m - 2hz_m), \\ [I + p/4H_x \bar{A}_{m+1}^*]u_{m+1} &= u_m - p/4[H_x \bar{f}_{m+1} - 2h(z_{m+1}^* + \bar{z}_{m+1})], \end{aligned}$$

where  $I$  is the unit matrix. This implicit scheme requires the inversion of a block tridiagonal matrix. A similar analysis for Eq. (1.2) gives the implicit scheme

$$(2.3) \quad \begin{aligned} u_{m+1}^* &= \hat{u}_m - p/2(A_m H_x u_m - 2hz_m), \\ [I + p/4A_{m+1}^* H_x]u_{m+1} &= [I - p/4\bar{A}_{m+1} H_x]u_m + ph(z_{m+1}^* + \bar{z}_{m+1})/2. \end{aligned}$$

**3. Explicit Two-Space Dimensional Scheme.** We now briefly mention the analogues of (1.7) and (1.8) for the equations

$$(3.1) \quad \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}(u, x, y, t) + \frac{\partial g}{\partial y}(u, x, y, t) = z(u, x, y, t),$$

$$(3.2) \quad \frac{\partial u}{\partial t} + A(u, x, y, t) \frac{\partial u}{\partial x} + B(u, x, y, t) \frac{\partial u}{\partial y} = z(u, x, y, t).$$

For (3.1) the analogue of (1.7) is

$$(3.3) \quad \begin{aligned} u_{m+1}^* &= \hat{u}_m - ap[H_x f_m + H_y g_m - 2hz_m], \\ u_{m+1} &= u_m - p/4 \left[ \left(1 - \frac{1}{2a}\right) \{H_x f_m + H_y g_m - 2hz_m\} \right. \\ &\quad \left. + \frac{1}{2a} \{H_x f_{m+1}^* + H_y g_{m+1}^* - 2hz_{m+1}^*\} \right. \\ &\quad \left. + \{H_x \bar{f}_{m+1} + H_y \bar{g}_{m+1} - 2h\bar{z}_{m+1}\} \right] \end{aligned}$$

whilst for (3.2) it takes the form

$$(3.4) \quad \begin{aligned} u_{m+1}^* &= \hat{u}_m - ap[A_m H_x u_m + B_m H_y u_m - 2hz_m], \\ u_{m+1} &= u_m - p/4 \left[ \left(1 - \frac{1}{2a}\right) \{A_m H_x u_m + B_m H_y u_m - 2hz_m\} \right. \\ &\quad \left. + \frac{1}{2a} \{A_{m+1}^* H_x u_{m+1}^* + B_{m+1}^* H_y u_{m+1}^* - 2hz_{m+1}^*\} \right. \\ &\quad \left. + \{\bar{A}_{m+1} H_x u_m + \bar{B}_{m+1} H_y u_m - 2h\bar{z}_{m+1}\} \right], \end{aligned}$$

where

$$u_m = u_{i,j}^m = u(ih, jh, mk),$$

$$H_x u_m = u_{i+1,j}^m - u_{i-1,j}^m, \quad H_y u_m = u_{i,j+1}^m - u_{i,j-1}^m$$

and

$$\hat{u}_m = \frac{1}{4}[u_{i+1,j}^m + u_{i-1,j}^m + u_{i,j+1}^m + u_{i,j-1}^m].$$

These schemes are again second-order correct replacements of the respective differential equations. As in the one-dimensional case, an iterative scheme can be developed in an obvious way. Also the new boundary technique can be applied, though the computational effort is now increased. These processes will not be described.

**4. Implicit Two-Dimensional Scheme.** In two dimensions the analogues of (2.2) and (2.3) are the alternating direction implicit schemes given, for (3.1), by

$$(4.1) \quad \begin{aligned} u_{m+1}^{**} &= \hat{u}_m - \frac{1}{2}p[H_x f_m + H_y g_m - 2hz_m], \\ [I + p/4H_y \tilde{B}_{m+1}^{**}]u_{m+1}^* &= u_m - p/4\{[H_x \tilde{f}_{m+1} + H_y \tilde{g}_{m+1}] - 2h(z_{m+1}^{**} + \bar{z}_{m+1})\} \\ &\quad + p^2/16H_y \tilde{B}_m H_x f_m, \\ [I + p/4H_x \tilde{A}_{m+1}^{**}]u_{m+1} &= u_{m+1}^*, \end{aligned}$$

where

$$f(u, x, y, t) = \tilde{A}(u, x, y, t) \cdot u, \quad g(u, x, y, t) = \tilde{B}(u, x, y, t) \cdot u$$

and a similar scheme for (3.2).

These A.D.I. schemes employ the D'jakonov splitting [1].

**5. Stability.** So far no discussion has taken place as to the stability characteristics of these schemes. The results of [3] have been extended in two ways. Firstly, the vectors  $f$  (and  $g$ ) have now been allowed to be functions of  $x, t, (y)$  in addition to  $u$ . Since our stability analysis in [3] was a linearized one, this extension does not affect the results. Secondly, we have allowed lower-order terms  $z$  in the equation. Since stability is a limiting process as  $h \rightarrow 0$  it can be seen from the difference schemes that in the limit, the vector functions  $z$  have no effect on stability. The stability characteristics of the schemes in this paper are therefore taken to be those of their counterparts in [3]. Therefore the implicit schemes are unconditionally stable, the one-dimensional explicit scheme is stable if  $a \geq \frac{1}{4}, p \leq 1/a^{1/2}$ , and the two-dimensional explicit scheme is conditionally stable (exact inequality only derivable for specific problems). Likewise the convergence of the iterative schemes depends on the results of [3].

It is not clear whether the new boundary technique will affect the stability characteristics of the explicit methods. In experiments no difficulty has been encountered. It should be noted that the new technique could also be applied at the "predictor" level in the implicit schemes.

**6. Numerical Results.** The above methods were tested on two problems. The first was the one-dimensional equation

$$\frac{\partial u}{\partial t} + \frac{x^2 u^2}{t} \frac{\partial u}{\partial x} = \left( \frac{2x^3 u^2}{t} - 1 \right) \cos (x^2 - t),$$

subject to the initial condition  $u(x, 1) = \sin (x^2 - 1)$  and the boundary condition  $u(0, t) = -\sin t$ . This problem has the theoretical solution

$$u(x, t) = \sin (x^2 - t)$$

and was solved in the region  $0 \leq x \leq 1, 1 \leq t \leq 1 + 100k$ , where  $k = ph$  is the step forward in the time direction. The space increment  $h$  was chosen to be 0.1 and  $p$  took the values 0.6 and 1.0.

TABLE 1

		<i>Explicit Methods</i>			<i>Implicit Methods</i>
<i>p</i>	<i>a</i>	<i>Theoretical Boundary</i>	<i>Two Iterations</i>	<i>Boundary Technique</i>	<i>Theoretical Boundary</i>
.6	.25	$.1277 \times 10^{-1}$	—	$.1997 \times 10^{-3}$	—
1.0	.25	$.1623 \times 10^{-1}$	—	$.1187 \times 10^{-2}$	—
.6	.50	$.7222 \times 10^{-2}$	$.4131 \times 10^{-2}$	$.3236 \times 10^{-2}$	$.2231 \times 10^{-3}$
1.0	.50	$.5367 \times 10^{-2}$	$.2339 \times 10^{-2}$	$.2559 \times 10^{-3}$	$.1629 \times 10^{-2}$

The errors are quoted in Table 1 for the point  $x = .7$  and are representative of the errors at  $t = 1 + 100k$ . The advantages of iterating the corrector or using the boundary technique are evident from the above table. However, the new boundary technique is considerably less time consuming than the iterative method.

The second problem considered was the two-dimensional equation

$$\begin{aligned} \partial u / \partial t + u e^{-t}(1+x+x^2-y) \partial u / \partial x + u e^{-t}(1+y+y^2-x) \partial u / \partial y \\ = -u[1+e^{-2t}(2+x^2+y^2)], \end{aligned}$$

subject to the conditions

$$\begin{aligned} u(x, y, 0) &= 1 - x - y \\ u(0, y, t) &= (1 - y)e^{-t}, \quad u(x, 0, t) = (1 - x)e^{-t}, \end{aligned}$$

which has the solution  $u = (1 - x - y)e^{-t}$ . The problem was solved in the region  $0 \leq x, y \leq 1, 0 \leq t \leq 100k$ , and the errors are quoted in Table 2.

TABLE 2

		<i>Explicit Methods</i>			<i>Implicit Methods</i>
<i>p</i>	<i>a</i>	<i>Theoretical Boundary</i>	<i>Two Iterations</i>	<i>Boundary Technique</i>	A.D.I.
0.6	.25	*	—	$-.4151 \times 10^{-4}$	—
1.0	.25	*	—	$-.1182 \times 10^{-1}$	—
0.6	.50	*	$-.1635 \times 10^{-5}$	$-.1175 \times 10^{-5}$	$-.3290 \times 10^{-5}$
1.0	.50	*	$-.5322 \times 10^{-6}$	$-.4392 \times 10^{-6}$	$-.1120 \times 10^{-6}$

In Table 2, the entries marked \* indicate that nonlinear instability had occurred prior to  $t = 100k$ . The above results show clearly the advantages of the two techniques for incorporating or replacing the extra boundary data.

**Conclusion.** The boundary replacement technique introduced above seems to have much to offer in the way of increased accuracy (and perhaps stability). The error distribution over a line (or plane) at a given time level tends to be "smoother" than if theoretical boundary data were employed. Although such smoothness may be obtained by iterating the corrector, it is a much more time consuming strategy.

The methods developed in this paper can be extended in a natural way to a higher number of space dimensions.

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